Math 3236 Statistical Theory 3/14/23

Last Time:

X_i uniform in $[0, A]$

A unknown

Found 2 estimators:

$$\hat{A}_1 = 2\bar{X} = \frac{2}{N}\sum_i X_i$$

$$\hat{A}_2 = \max_i X_i \qquad \hat{A}_3 = \left(1 + \frac{1}{N}\right)\max_i X_i$$

$\hat{A}_1$ Method of Moment Estimator

$$E(X_1) = m(A)$$

The MM estimator $\hat{A}$ solves

the equation

$$m(\hat{A}) = \bar{X}$$

For the uniform in $[0, A]$ we

have $\mathbb{E}(X_i) = m(A) = \frac{A}{2}$

$$\hat{A} = 2\bar{X}.$$

ML estimator.

$$f(\underline{x} \mid \theta) = \prod_{i=1}^{N} f(x_i \mid \theta)$$

joint p.d.f. of the X_i

likelihood function. when looked has a function of θ for fixed $\underline{x}$.

θ_{MLE} solves the equation

$$\hat{\theta}_{MLE}(X) = \sup_{\theta} f(\underline{X} \mid \theta)$$

In practice this means given $\underline{x}$ you find the θ that maximizes $f(\underline{x} \mid \theta)$ and then write it as a function of $\underline{X}$.

Assume X_i are exponential with par λ unknown.

$$f(x \mid \lambda) = \lambda e^{-\lambda x}$$

Sample size is N

$$L(\lambda) = f(\underline{x} \mid \lambda) = \prod_{i=1}^{N} f(x \mid \lambda) = \lambda^n e^{-\lambda \sum_i x_i}$$

$$\ell(\lambda) = \ln L(\lambda) = \sum_{i=1}^{N} \ln f(x \mid \lambda)$$

log-likelihood.

The λ that maximizes ℓ is the same that maximeze L.

$$L(\lambda) = \lambda^N e^{-\lambda \sum_i x_i}$$

$$L'(\lambda) = N \lambda^{N-1} e^{-\lambda \sum_i x_i} - \lambda^N \sum_i x_i \, e^{-\lambda \sum_i x_i}$$

$$L'(\lambda) = 0 \qquad N - \lambda \sum_i x_i = 0$$

$$\lambda(\underline{x}) = \frac{N}{\sum_i x_i}$$

$$\hat{\lambda}_{ML}(\underline{X}) = \frac{1}{\overline{X}}$$

$\hat{\lambda}_{ML}(\underline{X})$ is the same as the MoM estimator.

Is it a maximum?

$$L'(\lambda) = N\lambda^{N-1} e^{-\lambda \sum_i x_i} - \lambda^N \sum_i x_i \, e^{-\lambda \sum_i x_i}$$

$$L''(\lambda) = N(N-1)\lambda^{N-2} e^{-\lambda N\bar{x}} -$$

$$2N\lambda^{N-1} N\bar{x} \, e^{-\lambda N \bar{x}} +$$

$$\lambda^N N^2 \bar{x}^2 e^{-\lambda N \bar{x}}$$

$$\ell(\lambda) = N \log \lambda - \lambda \sum_{i=1}^{N} x_i$$

$$\ell'(\lambda) = \frac{N}{\lambda} - \sum_{i=1}^{N} x_i$$

$$\ell''(\lambda) = -\frac{N}{\lambda^2} \leq 0$$

$$L(\lambda) = \lambda^N e^{-\lambda \sum_i x_i}$$

$L(0) = 0$ $\quad$ $L(+\infty) = 0$

$L(\lambda) > 0$ $\quad$ if $\lambda \neq 0, \infty$

$\hat{\lambda}_{ML}$ is consistent

and it is asymptotically

Normal.

Likelihood function

$L(\theta)$

is not a probability

X_i is a Bernoulli r.v. p unknown

$$\underline{x} \qquad n(\underline{x}) = \# \text{ of } 1 \text{ in } \underline{x}$$

$$f(\underline{x} \mid p) = p^{n(\underline{x})} (1-p)^{N-n(\underline{x})}$$

$$\ell(p) = n(\underline{x}) \ln p + \left(N - n(\underline{x})\right) \ln(1-p)$$

$$\ell'(p) = \frac{n(\underline{x})}{p} - \frac{N - n(\underline{x})}{(1-p)}$$

$$p = \frac{n(\underline{x})}{N} \qquad n(\underline{x}) = \sum_{i=1}^{N} x_i$$

$$\hat{p}(X) = \bar{X}$$

M.L. and MoM give the same answer.

X_i have Γ distribution $(\alpha, 1)$

$$f(x|\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$$

$$f(\underline{x}|\alpha) = \frac{1}{\Gamma(\alpha)^N} \left(\prod_{i=1}^{N} x_i \right)^{\alpha-1} e^{-\sum_i x_i}$$

$$\ln f(\underline{x}|\alpha) = -N \, \Gamma(\alpha) + (\alpha-1) \sum_i \ln x_i - \sum_i x_i$$

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \frac{1}{N} \sum_i \ln x_i$$

$\ell(\alpha)$ $\quad \alpha_0$ such that $\ell(\alpha_0) > 0$

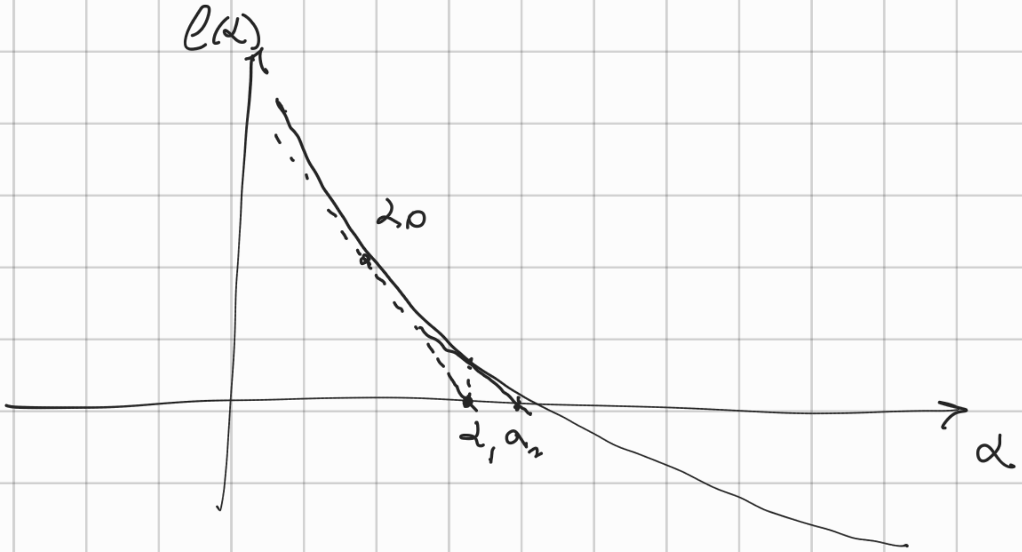
α_1 such that $\ell(\alpha_1) < 0$

$$\alpha_2 = \frac{\alpha_0 + \alpha_1}{2}$$

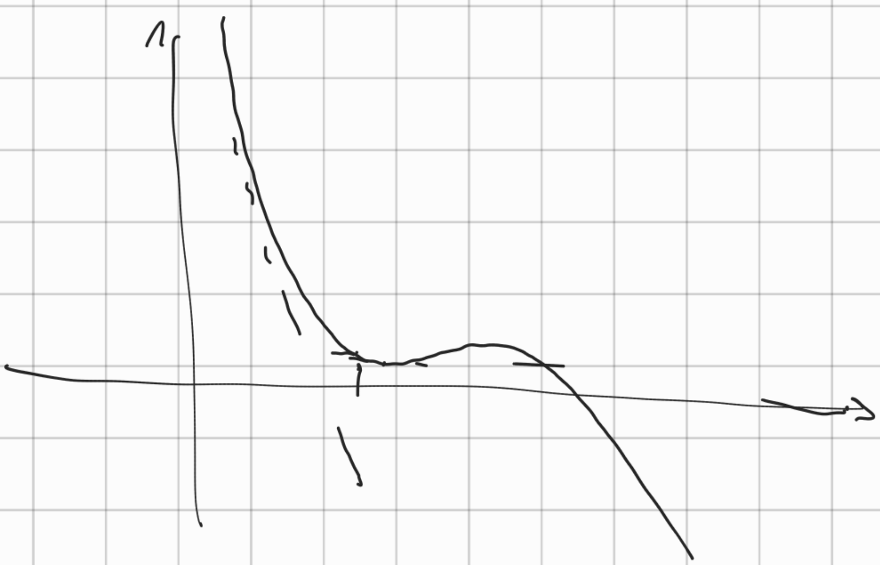
if $\ell(\alpha_2) > 0$ $\quad \alpha_0 \leftarrow \alpha_2$

if $\ell(\alpha_2) < 0$ $\quad \alpha_1 \leftarrow \alpha_2$

Newton Method.



$$\alpha_1 = \alpha_0 - \frac{\ell(\alpha_0)}{\ell'(\alpha_0)}$$



$\ell(\underline{\theta}) \qquad \underline{\theta} \in \mathbb{R}^2$

$\ell'(\underline{\theta}) = 0$

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$X_i \qquad f(x \mid \alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$

$\mathbb{E}(X_i) = \alpha$

$\hat{\alpha}(\underline{X}) = \bar{X}$.

$\circ$

$$f(x \mid \theta) = \frac{1}{\pi(1 + (x-\theta)^2)}$$

θ unknown

$$\ell(\theta) = \sum_{i=1}^{N} \ln(1 + (x_i - \theta)^2)$$

find maximum.

$\mathbb{E}(X_i) =$ does not exist.

$\mathbb{E}(\sqrt{|X_i|}) = m_{\frac{1}{2}}(\theta)$

$m_{\frac{1}{2}}(\theta) = \frac{1}{N} \sum_i \sqrt{|X_i|}$

Invariance principle.

$$f(x \mid \theta) \qquad \lambda = g(\theta)$$

assume that g is invertible

$$f(x \mid g^{-1}(\lambda)) = \tilde{f}(x \mid \lambda)$$

and then find

$$\hat{\lambda}_{ML}(\underline{X}) = g(\hat{\theta}_{ML}(\underline{X}))$$

Exp

$$f(x \mid \lambda) = \lambda e^{-\lambda x}$$

$$\tilde{f}(x \mid \mu) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

$$\hat{\lambda}_{ML} = \frac{1}{\overline{X}} \qquad \mu_{ML} = \overline{X}$$

$$\max_{\lambda} f(\underline{x} \mid \underline{g}^{-1}(\underline{\lambda}))$$